On Φ-Recurrent Lorentzian α-Sasakian manifold with semi symmetric Non metric connection

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Abstract

The present work deals with the study of Φ-recurrent Lorentzian α-Sasakian manifold with semi-symmetric non metric connection.

Keywords : Locally Φ-symmetric manifold Φ-recurrent Lorentzian a-Sasakian manifold, η η- Einstein manifold.

Introduction

The notion of local symmetry of a Riemannian manifold has been studied by many authors in several ways to a different extent. In 1977, Taka hasi[9] introduced the notion of locally Φ-symmetric Sasakian manifold and obtained their several interesting results. Generalizing the notion of Φ-symmetry , De, U.C[4] introduced the notion of Φ-recurrent Sasakian manifold.

Fridmann and Schouten introduced the idea of semi-symmetric linear connection on a differentiable manifold. Hayden introduced the idea of metric connection with torsion on Riemannian manifold. Yano[8], Golab [5] defined and studied semi-symmetric and quarter symmetric connection with affine connection . Further many authors like De,U.C.[1], Sharfudin and Hussain[3], Rastogi , Mishra and Pandey , Bagewadi and many other studied the various properties of semi- symmetric connection.

In this paper we study Φ-recurrent Lorentzian α-Sasakian manifold with semi-symmetric non metric connection and proved that a Φ-recurrent Lorentzian α-Sasakian manifold with symmetric non metric connection is a η η- Einstein manifold. Further we show that in Φ-recurrent Lorentzian α-Sasakian manifold with semi-symmetric non metric connection , the characteristic vector ξ and vector field η associated to the 1- form A are co-directional.

Preliminaries

A differentiable manifold M of dimension n is called a Lorentzian α sasakian

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manifold of it admints a tensor field Φ of type (1,1), the characteristic vector ξ , a covariant vector field η and lorentzian metric g which satisfy

$\Phi^2 \Phi^2 = 1 + \eta \otimes \xi$ (2.1)

$\eta(\xi) = -1$ (2.2)

$g(\Phi X, \Phi Y) = g(X,Y) + \eta(X)\,\eta(Y)$ (2.3)

$g(X, \xi) = \eta(X)$ (2.4)

$\Phi\,\xi = 0,\ \eta(\Phi X) = 0$ (2.5)

$(D_x D_x \Phi)Y = \alpha\, g(X,Y)\,\xi - \alpha\,\eta(Y)X$ (2.6)

for all $X, Y \in Tm$ [2,3,13]

Also a lorentzian α sasakian manifold m satisfies

$(D_x D_x\ \xi)Y = \alpha\,\Phi\,X$ (2.7)

$(D_x D_x\ \eta)Y = -\alpha\, g(\Phi X,Y)$ (2.8)

Where D denotes the operator of covariant differentiation with respect to lorentzian matric g.

Also on a Lorentzian α sasakian manifold , the following hold [2,3,13]

$R(X,Y)\,\xi = \alpha^2\alpha^2(\eta(Y)X - \eta(X)Y)$ (2.9)

$R(\xi, X)Y = \alpha^2\alpha^2\,(g(X,Y)\,\xi - \eta(Y)X)$ (2.10)

$R(\xi, X)\,\xi = \alpha^2\alpha^2\,(\eta(X)\,\xi + X)$ (2.11)

$S(X, \xi) = (n-1)\,\alpha^2\alpha^2\,\eta(X)$ (2.12)

$\eta(R(X, Y)Z) = \alpha^2\alpha^2\,(g(Y, Z)\,\eta(X) - g(X, Z)\,\eta(Y))$ (2.13)

$g(R(\xi, X)Y, \xi) = -\,\alpha^2\alpha^2\,[g(X,Y) + \eta(X)\,\eta(Y)]$(2.14)

For any vector field X,Y,Z where S is the Ricci curvature and Q is the Ricci operation given by

$S(X,Y) = g\,(\Phi X,Y)$

A lorentzian α sasakian manifold is said to be η- Einstein manifold if its Ricci tensor S takes the form

$S(X,Y) = a\, g(X,Y) + b\,\eta(X)\,\eta(Y)$

for arbitrary vector X,Y where a and b are function on M. If b=0 the η- Einstein manifold becomes Einstein manifold. [3,9] have proved that if Lorentzian α sasakian manifold M is η-

Einstein manifold then $a + b = -\alpha^2 - \alpha^2 (n-1)$.

Definition 2.1: A Lorentzian α sasakain manifold is said to be locally Φ- symmetric if

$$\Phi^2 \Phi^2 ((D_W D_W R)(X, Y)Z) = 0 \qquad (2.15)$$

Definition : 2.2

A Lorentzian α sasakian manifold is said to be recurrent if there exists a non zero 1-form A such that

$$\Phi^2 \Phi^2 ((D_W D_W R)(X, Y)Z) = A(W)R(X, Y)Z, \qquad (2.16)$$

Where A(W) is defined by A(W) =g(W, ρ) and ρ is a vector field associated with 1-from.

Lorentzian α sasakian manifold with semi symmetric non metric connection:

A semi symmetric connection $\overline{D}\,\overline{D}$ in Lorentzian α sasakian manifold can be defined by

$$\overline{D}_x \overline{D}_x Y = \overline{D}_x \overline{D}_x Y + \eta(Y)X \qquad (3.1)$$

Also we have $(\overline{D}_x \overline{D}_x g)(Y,Z) = -\eta(Y)g(Y,Z) - \eta(Z)g(Y,X) \qquad (3.2)$

A connection given by (3.1) with (3.2) is called semi symmetric non metric connection in Lorentzain α sasakian manifold.

A relation between curvature tensor M of the manifold with semi metric connection non metric connection $\overline{D}\,\overline{D}$ and Levi- Civita connection D is given by

$$\overline{R}\,\overline{R}(X,Y)Z = R(X, Y)Z - \alpha\, g(\Phi X, Z)Y - \alpha\, g(\Phi Y,Z)X \quad (3.3)$$

Where $\overline{R}\,\overline{R}$ and R are the Riemannian curvature of the connections $\overline{D}\,\overline{D}$ and D respectively.

From (3.3) , we have $\overline{S}\,\overline{S}(Y, Z) = S(Y, Z) + \alpha(n-1)\, g(\Phi Y, Z) \quad (3.4)$

Where $\overline{S}\,\overline{S}$ and S are the Ricci tensor of the connections $\overline{D}\,\overline{D}$ and D respectively.

Contracting (3.4), we get $\overline{r}\,\overline{r} = r \qquad (3.5)$

Where $\overline{r}\,\overline{r}$ and r are the scalar curvatures of the connections $\overline{D}\,\overline{D}$ and D respectively.

Φ- recurrent Lorentzian α sasakian manifold with semi symmetric non metric connection.

A analogous to the definition (2.2) we define a Lorentzian α sasakian manifold is said to be Φ - recurrent with respect to semi symmetric non metric connection if its curvature tensor $\overline{R}\,\overline{R}$ satifies the following condition

$\Phi^2 \Phi^2 (D_W D_W \overline{R}\,\overline{R})(X, Y) Z) = A(W)\, \overline{R} A(W)\, \overline{R}(X,Y)Z)$ (4.1)

using (2.1) in (4.1), we get

$(\overline{D}_W \overline{R}\, (\overline{D}_W \overline{R})(X, Y) Z + \eta((\overline{D}_W \overline{R}(\overline{D}_W \overline{R}))(\overline{D}_W \overline{R})(X,Y)Z)\xi = A(W)\, \overline{R}\,\overline{R}\,(X, Y)Z$ (4.2)

from which it follows that

$g((\overline{D}_W \overline{R}\, (\overline{D}_W \overline{R}))(X, Y)Z, U) + \eta((\overline{D}_W \overline{R}\, (\overline{D}_W \overline{R}))(X,Y)Z) g(\xi, U)$

$= A(W) g(\overline{R}\,\overline{R}(X, Y)Z, U)$ (4.3)

Let $\{e_1 e_1\}$, i = 1,2,3,...... n be an orthonormal basis of the tangent space at any point of the manifold. Then putting $X = U = \{e_1 e_1\}$ in (4.3) and taking summation over i, $1 \leq\leq i \leq\leq n$, we get

$((\overline{D}_W \overline{S}\, (\overline{D}_W \overline{S}))(Y,Z) + \eta((\overline{D}_W \overline{R}\, (\overline{D}_W \overline{R}))(e_1 e_1, Y)Z)\eta(e_1 e_1) = A(W)\overline{S}\,\overline{S}(Y, Z)$ (4.4)

putting $Z = \xi$,in (4.4), the second term of (4.4) takes the form

$g((\overline{D}_W \overline{R}\, (\overline{D}_W \overline{R}))(e_1 e_1, Y)\, \xi, \xi)$ which on simplification gives

$g((\overline{D}_W \overline{R}\, (\overline{D}_W \overline{R}))(e_1 e_1, Y)\, \xi, \xi) = 0$

Then from (4.4) we obtain

$(\overline{D}_W \overline{S}\, (\overline{D}_W \overline{S})\, (Y, \xi) = A(W)\, \overline{S}\,\overline{S}(Y, \xi)$ (4.5)

Now we know that

$(\overline{D}_W \overline{S}\, (\overline{D}_W \overline{S}))\, (Y, \xi) = \overline{D}_W \overline{S}\, \overline{D}_W \overline{S}\, (Y, \xi) - \overline{S}\, (\overline{D}_W \overline{S}(\overline{D}_w Y, \xi) - \overline{S}\,\overline{S}\, (Y, \overline{D}_W \xi\, \overline{D}_W \xi)$ (4.6)

Using (2.7), (2.8), (2.12), (3.4) in (4.6), we get

$(\overline{D}_W \overline{S}\, (\overline{D}_W \overline{S}))\, (Y, \xi) = \alpha S(Y, \Phi W) + S(Y, W) - \alpha(\alpha + 1)(n-1)\, g(Y, \Phi W) - \alpha^2\alpha^2 (n-1)\, g(Y,W) + \alpha^2 \alpha^2 (n-1)\, g(\Phi Y, \Phi W)$ (4.7)

In view of (4.5) and (4.7), we get

$\alpha S(Y, \Phi W) + S(Y, W) - \alpha(\alpha + 1)(n - 1) g(Y, \Phi W) - \alpha^2 \alpha^2 (n - 1) g(Y, W) + \alpha^2 \alpha^2 (n-1) g(\Phi Y, \Phi W) = \alpha^2 \alpha^2 (n-1) A(W) \eta(Y)$

Replacing $Y = \Phi Y$ in above equation, we get

$\alpha S(\Phi Y, \Phi W) + S(\Phi Y, W) - \alpha(\alpha + 1)(n - 1) g(\Phi Y, \Phi W) - \alpha^2 \alpha^2 (n - 1) g(\Phi Y, W)$

$+ \alpha^2 \alpha^2 (n - 1) g(Y, \Phi W) = 0$ (4.8)

Interchanging Y and W in (4.8) we get

$\alpha S(\Phi W, \Phi Y) + S(\Phi W, Y) - \alpha(\alpha + 1)(n - 1)g(\Phi W, \Phi Y) - \alpha^2 \alpha^2 (n - 1)g(\Phi W, Y)$

$+ \alpha^2 \alpha^2 (n - 1) g(W, \Phi Y) = 0$ (4.9)

Adding (4.8) and (4.9) and simplifying we get

$S(\Phi Y, \Phi W) = (\alpha^2 \alpha^2 + 1)(n-1)g(\Phi Y, \Phi W)$

Using (2.3) and (2.15), we get

$S(Y, W) = (\alpha^2 \alpha^2 + 1)(n-1)g(Y, W) + (n - 1) \eta(Y) \eta(W)$

This leads to the following theorem .

Theorem 4.1: A Φ - recurrent Lorentzian α sasakian manifold with semi symmetric non metric connection is η- Einstein manifold.

Again from (4.2), we have

$(\overline{D}_W \overline{R} (\overline{D}_W \overline{R}))(X, Y)Z = - \eta((\overline{D}_W \overline{R} (\overline{D}_W \overline{R}))(X, Y)Z) \xi \xi + A(W) \overline{R}\, \overline{R} (X, Y)$ (4.10)

From (2.13), (3.3) and using Bainchi identity we get

$A(W) \eta(\overline{R}\, \overline{R})(X, Y)Z) + A(X) \eta(\overline{R}\, \overline{R}(Y, W)Z) {}_{+} A(Y) \eta(\overline{R}\, \overline{R}(W, X)Z) = 0$ (4.11)

From(2.13), (3.3) in (4.11) we get

$A(W) \alpha^2 \alpha^2 [g(Y, Z) \eta(X) - g(X, Z) \eta(Y)] + A(X) \alpha^2 \alpha^2 [g(Z, W) \eta(Y) - g(Y, Z) \eta(W)] +$

$A(W)) \alpha^2 \alpha^2 [g(X,W) \eta(Z) - g(Z, W) \eta(X)] + \alpha[g(\Phi Y, Z) \eta(X) - g(\Phi X, Z) \eta(Y) +$

$g(\Phi W, Z) \eta(Y) - g(\Phi Y, Z) \eta(W) + g(\Phi X, Z) \eta(W) - g(\Phi W, Z) \eta(X)]= 0$ (4.12)

Putting Y = Z = e1e1 in (4.12) and taking summation over i, 1 ≤ ≤ i ≤ ≤ n,

we get $A(W) \eta(X) = A(X) \eta(W)$ (4.13)

For all vector fields, W. Replacing X by ξ ξ in (4.13), we get

$A(W) = - \eta (\rho) \eta(W)$ (4.14)

For any vector field W, where A(ξ ξ) = g(ξ ξ, ρρ) = η (ρ) , ρ ρ being vector field associated to the

1- form A that is g(X, ρ ρ) = A(X)

From (4.13) and (4.14) we state that following .

Theorem 4.2: In a Φ- recurrent Lorentzian α sasakian manifold with semi symmetric non metric connection the characteristic vector $\xi\xi$ and vector field $\rho\rho$ associated to the 1- form A are codirectional and 1- form A is given by (4.14).

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